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Evolution of Density Perturbations:

As we discussed, inflation can give rise to primordial perturbations with an almost scale-invariant spectrum. Now lets see how these perturbations evolve in time. We consider two different regimes;

(1) Superhorizon perturbations: Inflation generates inhomogeneities whose wavelength is larger than the horizon radius. During inflation physical wavelength grows exponentially, while horizon radius is almost constant. After inflation the wavelength grows $\propto t^{\frac{1}{2}}$ ($t^{\frac{2}{3}}$) in the radiation-dominated (matter-dominated) phase, while the horizon radius increases $\propto t$. In consequence, a given mode will eventually enter the horizon after a sufficiently long time.

The bottom line is that according to inflation all inhomogeneities start with their wavelengths being at superhorizon scales.

Density perturbations result in fluctuations in the geometry according to Einstein equations:

FRW Perturbed FRW

The homogeneous and isotropic FRW universe now becomes perturbed, which can be considered as ripples in the three dimensional hypersurfaces.

The important point is one can foliate the 4-dimensional spacetime in many ways (many choices for the 3-dimensional hypersurfaces). In the case of a homogeneous and isotropic universe one can the initial hypersurface as a homogeneous and isotropic spacelike hypersurface.

It will remain homogeneous and isotropic at later times too.

However, once inhomogeneities are introduced we do not have the privilege to start with a homogeneous and isotropic hypersurface that continues to be so later on. We can now choose any slicing of the spacetime.

This is a reflection of general relativity being invariant under general coordinate transformations, which implies general relativity being a gauge theory. No matter how we choose the hypersurfaces (that have both density perturbations and curvature perturbations in general), we must end up with the same result for $\frac{\delta T}{T}$ on the surface of last scattering (which is $O(10^{-5})$).

This is like any other gauge theory; one can choose any gauge to do calculations, but a physical quantity should have the same value independent of the chosen gauge.

In this case, we can choose any foliation, but the final result in temperature anisotropy of CMB as observed must be the same. We must therefore find gauge invariant quantities and calculate them. There is a number of these quantities, and they turn out to be constant while their characteristic wavelength is larger than the horizon size.

We do not discuss this in any further detail, and just point out that this is a very non trivial issue that requires careful treatment within the general relativity.

(2) Subhorizon perturbations. As mentioned, a given mode that is superhorizon at the end of inflation will eventually enter the horizon after a sufficiently long time. We now just have a perturbed fluid whose evolution is subject to causal dynamics. Lets consider an inhomogeneity with wavenumber "k":

$$\delta_k \rho \; e^{i\vec{k}\cdot\vec{r}} + \delta_k^* \rho \; e^{-i\vec{k}\cdot\vec{r}}$$

The fluid consists of baryons, photons and dark matter after $t \sim 1$ sec. The electrons are also there and interact with both photons and baryons (hence keeping them coupled to each other until recombination occurs), but we can ignore them because they are much lighter than the baryons.

Now consider the force on a tiny element of the fluid with volume dV. There are two contributions;

(a) Pressure gradient. This results in a force:

$$\boxed{F_{pressure} = -|\vec{\nabla} P|\, dV \quad k\,\omega\,\delta_k\,\rho\, dV}$$

Here ω relates pressure and energy density via the equation of state $P = \omega\rho$, and the $-$ve sign indicates that the pressure gradient tends to smooth the inhomogeneity. In a static background this just results in oscillating waves with a sound speed:

$$\boxed{v_s = \omega^{1/2}} \qquad \text{(in natural units)}$$

(b) Gravitational attraction. This results in a force:

$$\boxed{F_{gravity} \sim + \frac{G\,\delta_k\rho\;\rho\;\lambda^3}{\lambda^2}\, dV}$$

Here G is the Newton's gravitational constant, λ is the wavelength given by $\lambda = \frac{2\pi}{k}$, and ρ is the energy density in the matter (non relativistic particles). We have used the approximation that within half a wavelength

- the energy density is $\rho + \delta_k \rho$ (overdensity) and within the other half the energy density is $\rho - \delta_k \rho$ (underdensity). The positive sign indicates that gravity tends to amplify the inhomogeneity by attracting matter from the underdense to the overdense region.

Now the question is that in this competition between pressure and gravity which one wins. It is easy to see that pressur dominates if $\lambda < \lambda_J$, where;

$$\lambda_J \sim \frac{v_s}{(G\rho)^{1/2}}$$

is the Jean's length. On the other hand gravity is the dominant effect if $\lambda > \lambda_J$. In consequence:

- $\lambda_J < \lambda < H^{-1}$: inhomogeneities grow.

- $\lambda < \lambda_J$: inhomogeneities oscillate.

Next, we will derive the equations governing the dynamics of $\delta\rho_k$ in more detail and depth. We will see how the features in the CMB power spectrum (such as acoustic peaks) can be explained by combined action of pressure and gravity effects in a perturbed fluid. Eventually, we can use observation to determine cosmological parameters like the baryon density and dark matter density.